Nalanda Open University

M.sc Part-I

Course : Mathematics

Paper- V

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UNIT III

LINEAR ALGEBRA

Contents : Vector Space,

Linear Combination,

Finite Dimensional Vector Space,

Row And Column Space Of A Matrix,

Isomorphism,

Linear Transformation,

Dual Space And Dual Basis,

Projection

1. Vector Space, Vector Sub-Space, Linear Combination, Linear Dependence, Linear Independence

1.1. Introduction-:

It is physics which gives inspiration of the concept of a vector space as an algebraic system. It is a well known fact that in a plane or 3-dim. space vectors can be added, subtracted, and they can be multiplied by real or complex scalars.

Vector space is an algebraic generalization of the space of vectors. We often see their usefulness in solving the system of linear equations.

Linear algebra is a separate branch of algebra which combines the study of matrices and vector space.

1.2. Definitions-:

The basic idea of a group and field is the point of origin in the study of vector space.

Group:- A system (G, 0) containing a non empty set G and an operation 0 defined on it is called a group if the following conditions are satisfied:

(i) $a \, 0 \, b \in G \ \forall \ a, b \in G$ i.e, closure property is satisfied

(ii) $a \, 0 \, (b \, 0 \, c) = (a \, 0 \, b) \, 0 \, c$; $a, b, c \in G$ i.e, associative law is satisfied

(iii) There exists an element e in G such that $e \, 0 \, a = a \, 0 \, e = a \ \forall \ a \in G$
e is called the identity element of G. So existence of identity is satisfied.

(iv) For every a in G an element a^{-1} is in G such that
$a \, 0 \, a^{-1} = a^{-1} \, 0 \, a = e$. a^{-1} is called inverse of a.
However, if one additional proper that $a \, 0 \, b = b \, 0 \, a \ \forall \ a, b \in G$ also holds good then the group (G, 0) is called a commutative group or an abelian group.

Field:- A system (F, +, .) containing a non empty set F together with operation '+ '(addition) and ' .' multiplication is called a field if the following conditions are satisfied:

(1) (F, +) is an abelian (or commutative) group.

(2) (F, .) is an abelian group

(3) Multiplication is distributive w.r.t addition.

That is $a(b + c) = a \,.\, b + a \,.\, c$ and $(b + c) \,.\, a = b \,.\, a + c \,.\, a \ \forall \ a, b, c \in F$

Vector Space:- By K we shall understand either the set R of all real numbers or the set C of all complex numbers.

By a linear map we understand the following two maps.

(i) $(x, y) \rightarrow x + y$ from E x E into E called vector addition

(ii) $(\alpha, x) \rightarrow \alpha x$ from K x E into E called scalar multiplication.

The above two maps are assume to satisfy the following conditions:

(a) (E, +) is an abelian group

(b) $\alpha (x + y) = \alpha x + \alpha y$, $x, y \in E$, $\alpha \in K$.

(c) $(\alpha + \beta) x = \alpha x + \beta x$, $x \in E$, $\alpha, \beta \in K$

(d) $\alpha (\beta x) = (\alpha \beta) x$, $x \in E$, $\alpha, \beta \in K$

(e) $1 . x = x$, $x \in E$ and 1 is the unity element of K

Whenever the above conditions are satisfied we say that E is a vector space (or linear space) over the field K and in this case we write E(K). If there is no chance of confusion in we write simple E to mean that E is a vector space over some field K.

When K = R (the set of real numbers) we say E a real vector space

and if K = C (the set of complex numbers) we say E a complex vector space.

Vector Sub-Space:- Let W be a non empty subset of a vector space E(K) then W is called Vector Sub-Space of E(K) if W is itself a vector space over K under the operations defined over E.

Sum of two Vector Sub-Space :- Let w_1, w_2 be any two sub-spaces of a vector space E over the field K, then we define

$w_1 + w_2 = \{ x_1 + x_2 : x_1 \in w_1, x_2 \in w_2 \} = w$ (say)

it is easy to see that w is a vector space.

Direct sum of two vector sub-spaces :- A vector space E(K) is said to be direct sum of two vector sub-spaces w_1 and w_2 if

$E = w_1 + w_2$ and $w_1 \cap w_2 = \{0\}$

Whenever E is the direct sum of w_1 and w_2 , $E = w_1 \oplus w_2$.

Linear combination :- if $\{x_1, x_2, ...,x_n\}$ be a finite set of vectors of a vector space E(K) and $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ be a finite set of scalars of K then

$\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n$ is called the linear combinations (L . C) of vectors $(x_1, x_2, ...,x_n)$ and scalars $(\alpha_1, \alpha_2, ..., \alpha_n)$.

Finitely Generated Vector Space:- Let $\{x_1, x_2, x_3, \ldots, x_n\}$ be a non empty finite subset of a vector space E. Now if the sub-space $[x_1, x_2, \ldots, x_n]$ equals E then we say E is finitely generated.

Quotient Space:- Let U be a sub-space of a vector space E(K). Let $x \in U$ be arbitrary then

$x + U = \{ x + u : u \in U \}$ is called a co-set U in E.

Let E / U = the set of all co-sets of U in E = $\{ x + u : x \in E \}$

We define vector addition and scalar multiplication on the set E / U in the following ways:

(i) $(x + U) + (y + U) = (x + y) + U$
(ii) $\alpha (x + U) = \alpha x + U$, for every $x, y \in E$ and $\alpha \in K$.

Then E / U is a vector space called quotient Space of E by U.

Zero Vector :- Any vector is called a zero vector if each of its component is zero e.g $\{0, 0, 0, \ldots, 0\}$ is zero vector. The zero vector is denoted by 0 simply.

Linear Independence (L . I) :- A finite set $\{x_1, x_2, x_3, \ldots, x_n\}$ of elements of a vector space E(K) is called linearly independent set if

Whenever $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = 0$ for each $\alpha_i \in K$ then

$\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$

Linear Dependence (L . D) :- If $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = 0$ for $x_i \in E(K)$, $\alpha_i \in K$, for each i then some $\alpha_i \neq 0$. It is also called non trivial linear relation between $x_1, x_2, x_3, \ldots, x_n$.

Basis of a vector Space:- A non empty finite subset $\{x_1, x_2, x_3, \ldots, x_n\}$ of vectors of a vector space E(K) is said to be a basis of E if :

(i) The set $\{x_1, x_2, x_3, \ldots, x_n\}$ is linearly independent and
(ii) The sub-space $[x_1, x_2, \ldots, x_n]$, generated by vectors $x_1, x_2, x_3, \ldots, x_n$, equals E.

Equivalently:- A linearly independent set of vectors of a vector space E is called a basis of E if it generate E.

Dimension of a vector space:- The number of elements in a basis of a vector space E is called the dimension of E. The dimension of a vector space is also called the rank of it.

1.3. Theorem:

Theorem (1.3) i(2018) :- Any two bases of a vector space E have the same number of elements.

Proof :- Let $\{e_1, e_2, e_3, \ldots, e_n\}$ and $\{g_1, g_2, g_3, \ldots, g_n\}$ be any two bases of a vector space E(K).

Now since the set $\{e_1, e_2, e_3, \ldots, e_n\}$ of vectors is a basis

$\Rightarrow$ vectors $e_1, e_2, e_3, \ldots, e_n$ are linear independent.

Also $\{g_1, g_2, g_3, \ldots, g_n\}$ is a basis of E with m elements

Thus $n \leq m$ -----------------------(1)

Again the set of vectors $\{g_1, g_2, g_3, \ldots, g_n\}$ is a basis of E so its vectors are linearly independent.

Also $\{e_1, e_2, e_3, \ldots, e_n\}$ is a basis of E with n elements

Thus $m \leq n$ --------------------(2)

Thus from (1) and (2) $m = n$

That any two bases of a vector space have same number of elements.

Theorem (1.3) ii :- Let E be a vector space of dimension n. Then any set of n linearly independent elements of E is a basis of E.

Proof :- Let $\{x_1, x_2, x_3, \ldots, x_n\}$ be any set of n linearly independent elements of vector space E.

Then by a theorem for any x in E the set $\{x_1, x_2, x_3, \ldots, x_n, x\}$ of n + 1 elements of E is L . D but then we can get scalars $\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha$ not all zero such that

$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n + \alpha_n = 0$.

But by assumption $\{x_1, x_2, x_3, \ldots, x_n\}$ is linearly independent set.

Thus $\alpha \neq 0$

Thus $x = -\alpha^{-1} (\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n)$

Hence $\{x_1, x_2, x_3, \ldots, x_n\}$ generates E and is linearly independent.

Thus the set $\{x_1, x_2, x_3, \ldots, x_n\}$ forms a basis of E.

1.4. Solved examples :

Example 1:- The set of real numbers with usual definition of addition and multiplication is a field.

Example 2:- The set of complex numbers with usual definition of addition and multiplication is a field.

Example 3 (2018):- Show that vectors { x + 1, x – 1, -x + 5 } is linearly dependent.

Observation:- Let $e_1 = x + 1$; $e_2 = x - 1$; $e_3 = (-x + 5)$

Again let $a e_1 + b e_2 + c e_3 = 0$

Then $a (x + 1) + b (x - 1) + c (-x + 5) = 0$

$\Rightarrow x (a + b - c) + (a - b + 5 c) = 0$

$\Rightarrow a + b - c = 0$ ------- (i) and $a - b + 5 c = 0$ -------- (ii)

From (i) and (ii) we get

$a = - b + c$ and $a = b - 5c$

Solving these two we get $b = 3c$

Putting this value $b = 3c$ in (ii) we get $a = - 2c$

Thus these equations have a non trivial solution.

Also $a : b : c = - 2c : 3c : c$ i.e, $a : b : c = 2 : 3 : 1$

Thus the set of given vectors is linearly dependent.

Example 4(2018):- Show that the set $\{x^2 + 1, 3x - 1, -4x + 1\}$ is linearly independent.

Solution:- Let $e_1 = x^2 + 1$; $e_2 = 3x - 1$; $e_3 = -4x + 1$

Again let $a e_1 + b e_2 + c e_3 = 0$

$\Rightarrow a (x^2 + 1) + b (3x - 1) + c (-4x + 1) = 0$

$\Rightarrow x^2 (a) + x (3b - 4c) + (a - b + c) = 0$

Thus $a = 0$ ------- (i)

$3b - 4c = 0$ ------- (ii) and

$a - b + c = 0$ ------- (iii)

due to (i) and (iii) we get

$- b + c = 0 \Rightarrow b = c$ [since a = 0] ----------- (iv)

Thus from (iv) and (ii)

$3(c) - 4c = 0 \Rightarrow c = 0$

Thus $b = c = 0$

Thus $a = b = c = 0$

Thus $\{x^2 + 1, 3x - 1, -4x + 1\}$ is linearly independent.

Example 5 (2018):- Prove that the vectors (1, 0, -1), (1, 2, 1), (0, -3, -2) form a basis

Proof :- We prove this in two parts.

In first part we show that given vectors are linearly independent and in the second part we show that given vectors generate $V_3(R)$

For this first part : Let a (1, 0, -1) + b (1, 2, 1) + c (0, -3, -2) = 0 = (0, 0, 0)

Then $a + b = 0$ -------- (i) $\Rightarrow a = -b$ or $b = -a$

$2b - 3c = 0$ -------- (ii)

$-a + b - 2c = 0$ -------- (iii)

Since from (i) and (iii)

$b = c$ -------- (iv)

from (ii) and (iv)

$c = 0$

$\therefore b = c = 0 \Rightarrow b = 0$

But $b = -a \Rightarrow a = 0$

Thus we have

$a = b = c$

thus given vectors are linearly independent.

So the first part is done.

We now come to the 2nd part

For this, sufficient to show that any vector $x = (x_1, x_2, x_3)$ of $V_3(R)$ can be expressed as a linear combination

x = l (1, 0, -1) + m (1, 2, 1) + n (0, -3, -2) ----------- (A)

Then we have to determine l, m and n.

Since $x = (x_1, x_2, x_3) = (l + m, 2m – 3m, -l + m – 2n)$

Then $x_1 = l + m$ --------------- (i)

$x_2 = 2m – 3n$ --------------- (ii)

$x_3 = - l + m – 2n$ --------------- (iii)

from (i) and (iii)

$x_3 = - x_1 + m + m – 2n$

$\Rightarrow x_3 + x_1 = 2m – 2n = 2m – 3n + n = x_2 + n$ (from ii)

$\Rightarrow x_1 – x_2 + x_3 = n$

Also from (ii) $x_2 = 2m – 3 (x_1 – x_2 + x_3) \Rightarrow 2m = 3x_1 – 2x_2 + 3x_3 \Rightarrow m = ½ (3x_1 – 2x_2 + 3x_3)$

Again $l = x_1 – ½ (3x_1 – 2x_2 + 3x_3) = ½ [- x_1 + 2x_2 - 3x_3]$

Thus $x = (x_1, x_2, x_3) = l (1, 0, -1) + m (1, 2, 1) + n (0, -3, -2) = ½ [- x_1 + 2x_2 - 3x_3] (1, 0, -1)$

$+ ½ (3x_1 – 2x_2 + 3x_3) (1, 2, 1) + (x_1 – x_2 + x_3) (0, -3, -2)$

Hence we find that each vector of V_3 (R) is a linear combination of the given vectors.

It means the given vectors generate V_3 (R)

Therefore the given vectors (1, 0, -1), (1, 2, 1), (0, -3, -2) form a basis of V_3 (R).

Also the dimension of V_3 (R) is 3.

Example 6:- Prove that the vectors e_1= (1, 0, 0), e_2 = (0, 1, 0) and e_3 = (0, 0, 1) of V_3 (R) are linearly independent and they form a basis of V_3 (R). Also find the dim V_3 (R).

Proof :- Let $a_1 e_1 + a_2 e_2 + a_3 e_3 = 0$ where 0 is zero vector of V_3 (R) and $a_1, a_2, a_3 \in R$

$\Rightarrow a_1 (1, 0, 0) + a_2 (0, 1, 0) + a_3 (0, 0, 1) = (0, 0, 0)$ but then by definition scalar multiplication

$\Rightarrow (a_1, 0, 0) + (0, a_2, 0) + (0, 0, a_3) = (0, 0, 0)$

Thus by the definition of vector addition we have

$(a_1 + 0 +0, 0 + a_2 +0, 0 + 0 + a_3) = (0, 0, 0)$

$\Rightarrow (a_1, a_2 , a_3) = 0 \Rightarrow a_1 = a_2 = a_3 = 0$

Thus the set { e_1, e_2, e_3} of vectors of V_3 (R) is linearly independent subset of V_3 (R).

Also any vector x = (x_1, x_2, x_3) of V_3 (R) can be expressed as $x = x_1 e_1 + x_2 e_2 + x_3 e_3$

Thus each vector of V_3 (R) is a linear combination of e_1, e_2, e_3.

Hence V_3 (R) is generated by e_1, e_2, e_3.

Thus the set { e_1, e_2, e_3} is a basis of V_3 (R).

Also the dimension of V_3 (R) is 3 [no. of elements in basis { e_1, e_2, e_3}].

Note :- From above two examples it follows that a vector space may have more than one basis.

Example 7:- Let U be a sub-space of a vector space E(K) then the set E/U of all co-sets of U in E is a vector space under vector addition and scalar multiplication suitably defined as :

(1) $(x + U) + (y + U) = (x + y) + U$ and
(2) $\alpha (x + U) = \alpha x + U$, $x, y \in E$, $\alpha \in K$.

Proof :- By definition $E/U = \{ x + U : x \in E \}$

For $x, y \in E \Rightarrow x + y \in E \Rightarrow (x + y) + U \in E/U \Rightarrow (x + U) + (y + U) \in E/U$

Also $a \in K, x \in E \Rightarrow a x \in E \Rightarrow a x + U \in E/U \Rightarrow a (x + U) \in E/U$

That is $x + U, y + U \in E/U \Rightarrow (x + U) + (y + U) \in E/U$ [i.e; closure property is satisfied]

Also since $(x + U) + (y + U) = (x + y) + U = (y + x) + U = (y + U) + (x + U)$

(that is commutative law is satisfied)

Clearly $0 \in E \Rightarrow 0 + U \in E/U \Rightarrow 0$ is the additive identity of E/U

As $(x + U) + (0 + U) = (x + 0) + U = x + U$ [thus the existence of identity also holds good]

Clearly addition is associative also

Since for $x \in E \Rightarrow - x \in E$ as E is the vector space

Then $(- x + U) + (x + U) = (- x + x) + U = 0 + U$ (thus .....

Further as we have already seen above that for $a \in K$, $x + U \in E/U$ then

$a (x + U) \in E/U$

From which we find that

(1) $1 (x + U) = 1.x + U = x + U$
(2) $a. [b (x + U)] = (a b) (x + U)$
(3) it can also be easily verified that $(a + b) (x + U) = a (x + U) + b (x + U)$
(4) $a [(x + U) + (y + U)] = a (x + U) + a (y + U)$ can also be easily verified

Thus all the conditions for E/U to be a vector space are satisfied and hence E/U forms a vector space.

EXERCISES

1. A field K can be regarded as vector space over any satisfied F of K.
2. Show that vectors (3, 1, - 4), (2, 2, - 3), (0, - 4, 1) of V_3 (R) are linearly independent.
3. Prove that any subset of linearly independent set is linearly independent.
4. For what value of m the vector (m, 3, 1) is a linear combination of vectors $e_1 = (3, 2, 1)$, $e_2 = (2, 1, 0)$.
5. Show that the vectors (1, 1, - 1), (2, - 3, 5), (0, 1, 4) of R^3(R) are linearly independent.
6. Show that vectors (1, 0, 1), (1, 1, 1) and (0, 0, 1) of V_3(R) are linearly independent and they form a basis for V_3 (R).
7. Determine whether or not the following vectors form a basis of R^3 (1, 1, 2), (1, 2, 5), (5, 3, 4).

2. Finite Dimensional Vector Space, Quotient Space

2.1 Introduction:
In previous section we know what is the dimension of a vector space. In this section we shall know when we can say a vector space finite dimensional. Also we shall know about some of the properties of finite dimensional vector space (F.D.V.S).

2.2 Definition:

Linear span: Let S be a non empty subset of a vector space E(K) then the linear span of S is denoted by L(S) and is defined to be the set of all linear combinations of finite subset of the elements of S.

L(S) is also called the set generated by S.

Also if U be some other sub-space of E such that $S \subseteq U$ then $L(S) \subseteq U$.

We can conclude that L(S) is the smallest sub-space of E containing S.

Also $L(\phi) = \{0\}$. Clearly $L(S) = \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n : \alpha_1, \alpha_2, ..., \alpha_n \in K$, $x_1, x_2, x_3 ,.....,x_n$ are finite elements of S.

Finite Dimensional Vector Space: A vector space E(K) is said to be finite dimensional vector space (F.D.V.S) if there exists a finite S of U such that L(S) = E. In this case E is also said to be finitely generated.

Infinite Dimensional Vector Space: A vector space E(K) is called of infinite dimension if its dimension is not finite.

2.3 Theorem:-

Theorem (2.3) i :- If U is a sub-space of an n dimensional vector space E(K) then

$$\text{Dim. } U \leq n$$

Proof :- Let E(K) be a F.D.V.S with dimension n. Also let U be a sub-space of E.

Again let B = { $x_1, x_2, x_3, \ldots, x_n$ }be a basis of E.

Thus by definition L(B) = E $\Rightarrow$ each element of E is generated by a linear combination of elements of B.

Also B is a basis $\Rightarrow$ B is L.I

Thus either B is a basis of U or any subset of B is a basis of U.

Since every subset of L.I is L.I

Thus the basis of U, in no condition, will contain more than n vectors.

Hence Dim. U $\leq$ n (the number of the elements in the basis of E)

Theorem (2.3) ii :- If U is a sub-space of a finite dimension vector space E(K) then

$$\text{Dim. } E = \text{Dim. } U \text{ if and only if } E = U.$$

Proof :- U is a sub-space of F.D.V.S E(K) then U $\subset$ E.

Firstly let U = E. To prove that Dim. U = Dim. E

For this, E = U $\Rightarrow$ E is a sub-space of U, but U is a sub-space of E.

$$\Rightarrow \text{Dim. } E \leq \text{Dim. } U \text{ and Dim. } U \leq \text{Dim. } E \quad \Rightarrow \text{Dim. } U = \text{Dim. } E$$

Conversely let Dim. E = Dim. U. To prove E = U

Since Dim. E = Dim. U = n (say)

Let B = { $x_1, x_2, x_3, \ldots, x_n$ }be a basis of U so that B $\subset$ U, L(B) = U and B is L.I U $\subset$ E then $x_i \in U \Rightarrow x_i \in E \Rightarrow$ B is L.I subset of E

Also Dim. E = n $\Rightarrow$ every L.I subset of E containing n vectors is a basis of E.

Thus B is a basis of E $\Rightarrow$ L(B) = E

Hence U = L(B) = E

So U = E.

Theorem2.3 iii (2018):- Let W_1 and W_2 are two sub-spaces of a finite dimensional space E(K) then prove that

(i) $W_1 + W_2$ is finite dimensional

(ii) dim. W_1 + dim. W_2 = dim $(W_1 + W_2)$ + dim $(W_1 \cap W_2)$

Proof :- clearly $W_1 \cap W_2$ is a sub-space of E and dim. (E) is finite

Let dim. W_1 = m, dim. W_2 = n and dim $(W_1 \cap W_2)$ = r

Let $\{e_1, e_2, e_3, \ldots, e_r\}$ be the basis of $W_1 \cap W_2$.

Obviously this basis can be extended to the basis of W_1 and also to the basis of W_2.

Let $S_1 = \{e_1, e_2, e_3, \ldots, e_r, g_1, g_2, g_3, \ldots, g_{m-r}\}$ be the basis of W_1.

$S_2 = \{e_1, e_2, e_3, \ldots, e_r, p_1, p_2, p_3, \ldots, p_{m-r}\}$ be the basis of W_2.

We now set $S = \{e_1, e_2, e_3, \ldots, e_r, g_1, g_2, g_3, \ldots, g_{m-r}, p_1, p_2, p_3, \ldots, p_{n-r}\}$ -------------(1)

To see that S is a basis of $W_1 + W_2$

For this, sufficient to show that S is L.I and S spans $W_1 + W_2$.

For this, let $a_1e_1 + a_2e_2 + \ldots + a_r e_r + b_1 g_1 + b_2 g_2 + \ldots + b_{m-r} g_{m-r} + c_1p_1 + c_2p_2 + \ldots + c_{n-r} p_{n-r} = 0$

Where all $a_1, a_2 \ldots b_1, b_2 \ldots c_1, c_2 \ldots \in K$

Then $c_1p_1 + c_2p_2 + \ldots + c_{n-r} p_{n-r} = \sum ai\, ei + \sum bj\, gj \Rightarrow + \sum ck\, pk \in W_1$.

Similarly, $\sum ck\, pk \in W_2 \Rightarrow \sum ck\, pk \in (W_1 \cap W_2)$

Also $\sum ai\, ei + \sum bj\, gj = 0$ (since vectors $e_1, \ldots, e_r$ and $p_1, p_2, \ldots$ are L.I so $c_1 = c_2 = \ldots = c_{n-r} = 0$ and $a_1 + a_2 \ldots a_r = 0$)

Thus (1) is L.I.

We have to show that S spans $W_1 + W_2$

For this, Let e be any element of $W_1 + W_2$ then e = g + p (by definition)

Such that $g \in W_1$, $p \in W_2$. Also as S_1 and S_2 are bases of W_1 and W_2 respectively so g and p can be expressed as

g = $\sum_{i=1}^{r}$ ai ei + $\sum_{j=1}^{n-r}$ bj gi , a_i and b_j, answerable

and p = $\sum_{i=1}^{r}$ di ei +$\sum_{j=1}^{n-r}$ cj pj, d_i , $c_j \in K$

thus clearly e is the linear combination of elements of S.

Then S spans $W_1 + W_2$ i.e, S is basis for $W_1 + W_2 \Rightarrow W_1 + W_2$ is finite dimensional.

So the first part is done.

Further,

Also dimension of $W_1 + W_2 = m + n - r$ i.e, dim $(W_1 + W_2) = m + n - r$

But dim. W_1 + dim. $W_2 = m + n = r + (m + n - r) =$ dim $(W_1 \cap W_2)$ + dim $(W_1 + W_2)$

Or equivalently dim $(W_1 + W_2)$ = dim. W_1 + dim. W_2 - dim $(W_1 \cap W_2)$ proved.

Theorem (2.3) iv :-Let

(i) V(K) be a finite dimensional vector space
(ii) (ii) W_1 , W_2 be two sub-spaces of V(K)
(iii) V is the direct sum of W_1 and W_2

Then dim V = dim. W_1 + dim. W_2

Proof :- Since by hypothesis V is finite dimensional, W_1 , W_2 are its sub-spaces

Thus W_1 and W_2 are also finite dimensional

Let = dim. $W_1 = m$ and dim. $W_2 = n$.

Also given $V = W_1 \oplus W_2 \Rightarrow V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$

Let $S_1 = \{e_1, e_2, e_3 ,.....,e_m\}$ be the basis of W_1 .

$S_2 = \{ g_1, g_2, g_3 ,.....,g_n\}$ be the basis of W_2 .

let $S_3 = \{e_1, e_2, e_3 ,.....,e_m, g_1, g_2, g_3 ,.....,g_n\}$

then we find that:

$(a_1e_1 + a_2e_2 + ...+ a_m e_m) + (b_1 g_1 + b_2 g_2 +....+b_n g_n)$

$\Rightarrow (b_1 g_1 + b_2 g_2 +....+b_n g_n) = - (a_1e_1 + a_2e_2 + ...+ a_m e_m)$

Thus $a_1e_1 + a_2e_2 + ...+ a_m e_m \in W_1 \cap W_2$ and $b_1 g_1 + b_2 g_2 +....+b_n g_n \in W_1 \cap W_2$

But $W_1 \cap W_2 = \{0\} \Rightarrow$ both of the above L.C. are equal to zero

But S_1 and S_2 being bases are L.I. so all scalar are zero.

Thus S_3 is also L.I.

Let $t \in V$ be arbitrary then $t = e + g$, $e \in W_1$, $g \in W_2$

$\Rightarrow$ e, g can be expressed as L.C. with the elements of W_1 and W_2 separately

Thus $t = e + g = a_1e_1 + a_2e_2 + ...+ a_m e_m + b_1 g_1 + b_2 g_2 +....+b_n g_n$.

$\Rightarrow S_3$ generates V thus S_3 forms a basis of V.

Hence dim V = m + n = dim. W_1 + dim. W_2 proved.

Theorem (2.3) v (2017) :- Let W be a sub-space of a F.D.V.S. V then

dim. V/W = dim. V – dim. W

Proof :- Let dim. W = m and $\{ w_1, w_2, w_3 ,.....,w_n \}$ be a basis of W

$\Rightarrow (w_1, w_2, w_3 ,.....,w_n)$ is in W

$\Rightarrow$ it is L.I. in V then as we know,

$\{ w_1, w_2, w_3 ,.....,w_m, v_1, v_2, v_3 ,.....,v_n \}$ can be extended basis of V.

Thus dim. V = n + m

We consider the set $\{w + v_1, w + v_2, ... , w + v_n,\}$, we show it forms a bases of V/W .

Let $\alpha_1(w + v_1) + \alpha_2(w + v_2) + ... + \alpha_n(w + v_n) = w$, $\alpha_i \in K$ (the field)

$\Rightarrow W + (\alpha_1v_1 + \alpha_2v_2 + ...+ \alpha_n v_n) = W \Rightarrow \alpha_1v_1 + \alpha_2v_2 + ...+ \alpha_n v_n \in W$

$\Rightarrow \alpha_1v_1 + \alpha_2v_2 + ...+ \alpha_n v_n$ is L.C. of $w_1, w_2, w_3 ,.....,w_m$ =

$\Rightarrow \alpha_1v_1 + \alpha_2v_2 + ...+ \alpha_n v_n = \beta_1w_1 + \beta_2w_2 + ...+ \beta_m w_m$, $\beta_j \in F$

$\Rightarrow \alpha_1v_1 + \alpha_2v_2 + ...+ \alpha_n v_n - \beta_1w_1 - \beta_2w_2 - .....- \beta_m w_m = 0 \Rightarrow \alpha_i = \beta_j = 0$ for all i, j

$\Rightarrow \{ w + v_1, w + v_2, ... , w + v_n,\}$ is L.I.

Again, for any $w + v \in V/W$, $v \in V \Rightarrow v$ is linear combination of $w_1,.....,w_m, v_1, ...,v_n$.

Let $v = \alpha_1w_1 + \alpha_2w_2 + ...+ \alpha_m w_m + \beta_1v_1 + \beta_2v_2 + ...+ \beta_n v_n$ $\alpha_i, \beta_j \in K$

Again $w + v = W + (\alpha_1w_1 + \alpha_2w_2 + ...+ \alpha_m w_m) + (\beta_1v_1 + \beta_2v_2 + ...+ \beta_n v_n)$

$= W + (\beta_1 v_1 + \beta_2 v_2 + ...+ \beta_n v_n)$

$= \{ (w + \beta_1 v_1) + (w + \beta_2 v_2) +....+ (w + \beta_n v_n) \}$

$= \beta_1 (w + v_1) + \beta_2 (w + v_2) +....+ \beta_n (w + v_n)$

Thus S spans V/W and is therefore a basis is

$\Rightarrow$ dim. V/W = n

Therefore dim. V/W = dim. V – dim. W

2.4. Exercise:

Problem 1. The linear span L(S) of any non-empty subset S of a vector space V(F) is a subspace of V(F).

Problem 2. If S, T are two subsets of a vector space V, then

(1) $S \subseteq T \Rightarrow L(S) \subseteq L(T)$
(2) $L(S \cup T) = L(S) + L(T)$

Problem 3. The linear sum of two subspaces W_1 and W_2 of vector space V(F) is generated by their union.

That is $W_1 + W_2 = L (W_1 \cup W_2)$.

3. Row space and column space of a matrix, Dimension and Rank

3.1 Introduction: In this section we shall learn how to introduce the notion Linear combination, linearly independent etc. for the cases of matrix. We are already well acquainted with row and column but now we shall also know what are row and column spaces.

3.2 Definition:

Echelon matrix: Let A = [a_{ij}] be an $m \times n$ matrix over some field F. Now if the number zeros preceding the non zero elements of a row increases row by row. The elements of the last row or rows may be all zero then the matrix A is called an echelon matrix. Sometime we also call echelon form.

Distinguished elements of a matrix A: The first non zero elements in the rows of an echelon matrix A are called distinguished elements of A.

Row canonical form of a matrix : If distinguished elements are each equal to 1 and are the only non-zero elements in their respective columns. It is also called row reduced echelon matrix.

Example 1. Matrix $A = \begin{pmatrix} -3 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{pmatrix}$ is an echelon matrix.

Distinguished elements are – 3, 1, 4

Example 2. $A = \begin{pmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is a row reduced echelon matrix.

Row–Equivalence of two matrices: Let A and B be any two matrices then A is called row equivalent to B if and only if B can be obtained from A by a finite number of elementary row operations.

Column Equivalence of two matrices: A is called column equivalent to B iff B can be obtained from A by a finite number of elementary row operations.

Row space of a matrix: Let A = [a_{ij}] be an m × n matrix over a field F. Then the m vectors of rows of A are as below:

$R_1 = (a_{11}, a_{12}, \ldots, a_{1n})$ is an n tuple over F

$R_2 = (a_{21}, a_{22}, \ldots, a_{2n})$ is an n tuple over F

$R_m = (a_{m1}, a_{m2}, \ldots, a_{mn})$ is an n tuple over F

Then $L(R_1, R_2, \ldots R_m)$, the linear span, is a sub-space of F^n, is called the row space of A. Vectors $R_1, R_2, \ldots R_m$ are called row vectors. Row space A is denoted by R(A)

Column space of a matrix: Let A = [a_{ij}] be an m × n matrix over a field F.

Then $C_1 = (a_{11}, a_{21}, \ldots, a_{m1})$

$C_2 = (a_{12}, a_{22}, \ldots, a_{m2})$

$C_n = (a_{1n}, a_{2n}, \ldots, a_{mn})$

The linear span $L(C_1, C_2, \ldots C_n)$ is a sub-space F^m and we call it column space of A. generally we denote it by C(A). vectors $C_1, C_2, \ldots C_n$ are called column vectors.

Null space of a matrix A: Let A = [a_{ij}] be an m × n matrix over a field F. then we denote null space of A by N(A) and we define it as

$N(A) = \{ x \in F^n : A x = 0 \}$.

Note 1: column space of A is the same as the row space of A^t.

Row rank (or column rank) of a matrix A: The dim. R(A) is called row rank and

dim. C(A) is called column rank.

Also dim. R(A) = row rank of A = the number of non zero rows in the echelon matrix of A.

Rank of a matrix: Row rank of A (or column rank) of A is called the rank of A.

3.3. Theorem:

Theorem (3.3) i :- Row equivalent matrices have the same row space.

Proof :- Let A and B be any two row equivalent matrices.

Then by definition,

Each row of B is either a row of A or is the linear combination of rows of A.

Hence row space of B is contained in row space of A --------------(1)

On the other hand in a similar way we start from B and apply elementary inverse operations we can find that,

row space of A is contained in the row space of B -----------------(2)

Thus from (1) and (2) it follows that

The row space of A and the row space of B are the same. ,,

Theorem (3.3) ii :- Row space and column space of a matrix A have the same dimension

Or

Let A_{mxn} be a matrix then dim. of its row space is equal to the dim. of its column space.

Proof :- Let$\{ v_1, v_2, v_3, \ldots, v_k \}$be a basis for C(A)

Then each column of A can be expressed as a linear combination of these vectors,

We suppose that the i^{th} column C_i is given by

$C_i = \alpha_{1i}v_1 + \alpha_{2i}v_2 + \ldots + \alpha_{ni} v_k$

Let us form two matrices as follows:

B is an $m \times k$ matrix whose columns are the basis vectors v_i , while $C = (\alpha_{ij})$ is a $k \times n$ matrix whose i^{th} column contains the coefficient α_{1i} , α_{2i} , ..., α_{ki} then it follows that $A = BC$.

However we can also view the product $A = BC$ as expressing the rows of A as a linear combination of the rows C with the i^{th} row of B.

Now giving the coefficients for the linear combination that determine the i^{th} row of A.

Therefore the row of C are a spanning set for the row space of A, and so the dimension of the row space of A is almost K.

Thus we conclude that dim{ row space (A) } $\leq$ dim{ column space (A) }------------(1)

Applying the same argument to A^t we can see that

dim C(A) $\leq$ dim R(A) -----------------------(2)

thus it direct follows from (1) and (2) that

dim { C(A) }= { dim R(A) },,

Theorem (3.3) iii :- The non zero rows of an echelon matrix are linearly independent

Proof :- Let $R_1, R_2, \ldots\ldots\ldots R_n$ be the non zero rows of an echelon matrix A.

To prove $R_1, R_2, \ldots\ldots\ldots R_n$ are linearly independent vectors.

If not then let $R_1, R_2, \ldots\ldots\ldots R_n$ are linearly dependent

Then one of rows say R_m, is a linear combination of the preceding rows

That is, $R_m = \alpha_{m+1}R_{m+1} + \alpha_{m+2}R_{m+2} + \ldots + \alpha_n R_n$ --------------- (1)

Let k^{th} element of R_m be its first non zero entry.

Since matrix A is in echelon form, the k^{th} element of each of R_{m+1}, R_{m+2} , ...,R_n is zero.

Thus from (1), the k^{th} element of R_m .

= the k^{th} element of $\alpha_{m+1}R_{m+1} + \alpha_{m+2}R_{m+2} + \ldots + \alpha_n R_n$.

$= \alpha_{m+1} . 0 + \alpha_{m+2}. 0 + \ldots + \alpha_n . 0 = 0$ is a contradiction as by assumption k^{th} element of R_m is non zero.

Thus $R_1, R_2, \ldots\ldots\ldots R_n$ are linearly independent ,,

3.4 Solved example:

Example 1. Reduce $A = \begin{pmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{pmatrix}$ to echelon form and then to row reduced echelon form.

Solution : Operating $R_2 \rightarrow R_2 + (-2)R_1$, $R_3 \rightarrow R_3 + (-3)R_1$

Thus

$$A \sim \begin{pmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -4 & 4 \\ 0 & 7 & -7 & 6 \end{pmatrix}$$ operate $R_3 \rightarrow 3R_3 + (-7)R_2$ then we have

$$\sim \begin{pmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -4 & 4 \\ 0 & 0 & 7 & -10 \end{pmatrix}$$ which is an echelon form of A.

Now we reduce it to row reduced echelon form.

We operate $R_2 \rightarrow 1/3\, R_2$, $R_3 \rightarrow 1/7\, R_3$ then

$$A \sim \begin{pmatrix} 1 & -2 & 3 & -1 \\ 0 & 1 & -4/3 & 4/3 \\ 0 & 0 & 1 & -10/7 \end{pmatrix}$$

Now operating $R_1 \rightarrow R_1 + 2R_2$, then

$$A \sim \begin{pmatrix} 1 & 0 & 1/3 & 5/3 \\ 0 & 1 & -4/3 & 4/3 \\ 0 & 0 & 1 & -10/7 \end{pmatrix}$$

We now operate $R_2 \rightarrow R_2 + 4/3R_3$ and $R_1 \rightarrow R_1 + (-1/3)R_3$

$$\Rightarrow A \sim \begin{pmatrix} 1 & 0 & 0 & 15/7 \\ 0 & 1 & 0 & 4/3 \\ 0 & 0 & 1 & -10/7 \end{pmatrix}$$

This is the required form.

Example 2. Show that the matrices A and B have the same column space where

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 1 & 4 & 3 \\ 1 & 1 & 9 \end{pmatrix} \quad \text{and } B = \begin{pmatrix} 1 & 2 & 3 \\ -2 & -3 & -4 \\ 7 & 12 & 17 \end{pmatrix}$$

Solution: Matrix A and B will have the same column space if and only if A^t and B^t have some row space.

Thus we have to reduce A^t and B^t to row reduced echelon form:

$$\text{For, since } A^t = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 4 & 1 \\ 5 & 3 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & -2 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{And } B^t = \begin{pmatrix} 1 & -2 & 7 \\ 2 & -3 & 12 \\ 3 & -4 & 17 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 7 \\ 0 & 1 & -2 \\ 0 & 1 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 7 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} .$$

Thus from above it is clear that

The non zero rows in row reduced echelon matrix of A^t

= the non zero rows in row reduced echelon matrix of B^t

$\therefore R(A^t) = R(B^t) \Rightarrow C(A) = C(B)$ //

Example 3. Find the basis for the row space of the matrix A and determine the rank where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 1 & 1 & 5 \end{pmatrix}$$

4. Isomorphism

4.1 Introduction:
We have much studied about isomorphism in group theory. In this section we shall study the role of isomorphism in the context of a vector space and its field.

4.2 Definition:

Isomorphism of vector spaces: Let E and E′ be any two vector spaces over the same field F. Again Let $f : E \to E'$ be a mapping then f is called an isomorphism if

(1) f is one – one onto
(2) $f(x + y) = f(x) + f(y)$, $x, y \in E$, $f(x), f(y) \in E'$ and
(3) $f(\alpha x) = \alpha\ f(x)$, $\alpha \in F$, $x \in E$.

Whenever f is an isomorphism of E and E′, we say that E is isomorphic to E′. We also say that E′ is an isomorphic image of E.

4.3 Theorem:

Theorem (4.3) i :- Every n-dimensional vector space V(F) is isomorphic to $F^n(F)$

Proof :- By question V(F) is an n-dimensional vector space.

Let $S = \{ e_1, e_2, \ldots, e_n\}$ be a basis of V.

Then every vector of V can be expressed as linear combination of the elements of S.

Then for any $e \in V$, $e = a_1e_1 + a_2e_2 + \ldots + a_n e_n$, $a_1, a_2, \ldots, a_n \in F$.

Let $T : V \to F^n$ be a mapping given by,

$T(e) = T(a_1e_1 + a_2e_2 + \ldots + a_n e_n) = (a_1, a_2, \ldots, a_n)$ for every $e \in V$.

Then by uniqueness of the representation of e in the form $e = a_1e_1 + \ldots + a_n e_n$ the mapping T is well defined.

Also let $e, e' \in V$ and $a \in F$ then $e = a_1e_1 + \ldots + a_n e_n$, $e' = b_1e_1 + \ldots + b_n e_n$ where $a_i, b_i \in F$, $i = 1, 2, \ldots, n$.

To prove that:

$T(e + e') = T\{ (a_1 + b_1) e_1 + (a_2 + b_2) e_2 + \ldots + (a_n + b_n) e_n \}$

$= a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n = (a_1, a_2, a_3, \ldots, a_n) + (b_1, b_2, b_3, \ldots, b_n) = T(e) + T(e')$

Also $T(ae) = T\{a (a_1e_1 + a_2e_2 + \ldots + a_n e_n)\} = T\{ (a a_1) e_1 + (a a_2) e_2 + \ldots + (a a_n) e_n \}$

$= \{ a a_1, a a_2, ... , a a_n\} = a (a_1, a_2, a_3 ,.....,a_n) = a T(e)$

We also find that : $T(e) = T(e') \Rightarrow T (a_1e_1 + a_2e_2 + ...+ a_n e_n) = T (b_1e_1 + b_2e_2 + ...+ b_n e_n)$

$\Rightarrow = (a_1, a_2, a_3 ,.....,a_n) = (b_1, b_2, b_3 ,.....,b_n) \Rightarrow a_i = b_i$ for each i

$\Rightarrow e = e' \Rightarrow$ T is one-one

Also let any $(a_1, a_2, a_3 ,.....,a_n) \in F^n \Rightarrow a_1e_1 + a_2e_2 + ...+ a_n e_n \in V$

and $T(a_1e_1 + a_2e_2 + ...+ a_n e_n) = (a_1, a_2, a_3 ,.....,a_n) \Rightarrow$ T is onto

Thus all the conditions are satisfied for T to be isomorphism

Thus V(F) is isomorphic to F^n

Theorem (4.3) ii :- Two finite dimensional vector space over the same field F are isomorphic iff they are same dimension.

Proof :- Let E and E′ be any two isomorphic vector space over the field F.

Let $T : E \rightarrow E'$ be the isomorphism

Let dim. E = n and $\{ e_1, e_2,.....,e_n\}$ be a basis of E

We claim $\{ T (e_1), T (e_2), ... , T (e_n) \}$ is a basis of E′

Let $\sum^n_{i=1} a_i T (e_i) = 0, a_i \in F$

$= \sum T (a_i e_i) = 0 = T(0) \Rightarrow \sum a_i e_i = 0$ [since T is one-one]

$\Rightarrow a_i = 0 \ \forall$ i as $e_1, e_2,.....,e_n$ are linearly independent

$\Rightarrow T (e_1), T (e_2), ... , T (e_n)$ are linearly independent

Again if $g \in E'$ is any element, then as T is onto, $\exists$ some e in E such that

$$T(e) = g$$

Now $e \in E \Rightarrow a_1e_1 + a_2e_2 + ...+ a_n e_n, a_i \in F$ for each i = 1, 2, ..., n

$\Rightarrow g = T (e) = T (a_1e_1 + a_2e_2 + ...+ a_n e_n)$

$\Rightarrow$ g = a linear combination of $T (e_1), T (e_2), ... , T (e_n)$

Thus $T (e_1), T (e_2), ... , T (e_n)$ span E′

Thus $T (e_1), T (e_2), ... , T (e_n)$ forms a basis for E′

It makes clear that dim. E′ = n

Thus dim. E = dim. E′ = n provide E and E′ are isomorphic

Conversely:- Let dim. E = dim. E′

To prove E and E′ are isomorphic

For this, let { $e_1, e_2, \ldots, e_n$} be a basis of E and { $g_1, g_2, \ldots, g_n$} be a basis of E′.

Let T : E → E′ be a mapping given by

$T(e) = T(a_1e_1 + a_2e_2 + \ldots + a_n e_n) = a_1e_1 + a_2e_2 + \ldots + a_n e_n$.

Clearly T is well defined.

Also for any e, e′ ∈ E we see that:

$T(e + e') = T\{ (a_1e_1 + a_2e_2 + \ldots + a_n e_n) + (b_1e_1 + b_2e_2 + \ldots + b_n e_n) \}$

$= T\{ (a_1 + b_1) e_1 + (a_2 + b_2) e_2 + \ldots + (a_n + b_n) e_n \}$

$= (a_1 + b_1) g_1 + (a_2 + b_2) g_2 + \ldots + (a_n + b_n) g_n$

$= (a_1g_1 + a_2g_2 + \ldots + a_n g_n) + (b_1g_1 + b_2g_2 + \ldots + b_n g_n) = T(e) + T(e')$

That is $T(e + e') = T(e) + T(e')$

Also $T(ae) = T\{a (a_1e_1 + a_2e_2 + \ldots + a_n e_n)\} = T [a a_i e_i = [(a a_i) g_i$

$= a [a_i g_i = a T(e)$

We also see that :

If $e \in \ker T$ then $T(e) = 0 \Rightarrow T ([a_i e_i) = 0 \Rightarrow [a_i g_i = 0$

$\Rightarrow a_i = 0, \forall i, \; g_1, g_2, \ldots, g_n$ being linearly independent

$\Rightarrow e = 0$

$\Rightarrow \ker T = \{0\}$

⇒ T is one-one

Also T is clearly onto

Thus T is isomorphism.

Theorem (4.3) iii :- The complex plane is isomorphic to the Euclidean plane.

Proof :- Let V = c, the vector space of all complex numbers over the field R of all real numbers

Also let $v' = R^2$, the vector space of all ordered pairs of reals over the field R.

Let a mapping $T : V \to V'$ be defined as T (a + i b) = (a, b).

We see that:

T { (a + i b) + (c + i d) } = T { (a + c) + i (b + d) }= (a + c, b + d) = (a, b) + (c, d)

= T (a + i b) + T (c + i d)

Also T { α (a + i b) } = T (α a + i α b) = (α a, αb) = α (a, b) = α T (a + i b)

Also T (a + i b) = - T (c + i d)

Then T (a + i b - c – id) = 0 but T (a + i b - c – id) = T {(a - c) + i (b - d)} = (a - c, b - d)

Thus (a - c, b - d) = (0, 0) $\Rightarrow$ a – c = 0 and b – d = 0

$\Rightarrow$ a = c, b = d $\Rightarrow$ a + i b = c + i d

$\Rightarrow$ T is one-one

Also, obviously T is onto

Thus T is isomorphism of c onto R^2

5. Linear Transformation, Linear functional

5.1. Introduction:
It a kind of mapping (or transformation) which we shall study in linear spaces.

5.2. Definitions:

Linear Transformation: Let V and V′ be any two vector spaces over the field F. Then a mapping $T : V \to V'$ is called a linear transformation (L.T) if the following conditions are satisfied:

(i) $T(u + v) = T(u) + T(v), u, v \in V$

(ii) $T(\alpha u) = \alpha T(u), \alpha \in F, u \in V$

Conditions (i) and (ii) can be expressed together as

$T (\alpha u + \beta v) = \alpha T(u) + \beta T(v), u, v \in V, \alpha, \beta \in F$.

A linear transformation is also called a linear mapping.

Kernel of Linear Transformation (or Null space) : Let T be a linear transformation of a vector space V into a vector space V′. Then the kernel of T is denoted by ker T and we defined it as:

ker T = { $x \in V : T(x) = 0$ }.

ker T is also called Null space of T.

Kernel of Identity Transformation : It is denoted by $I : V \to V$ defined by the set { $x \in V : I(x) = x = 0$ } = {0}.

Kernel of Zero Transformation : It is defined by $0 : V \to V$ is { $x \in V : 0(x) = 0$ }= V.

Nullity of Linear Transformation: Let T be linear transformation of V into V′ then dimension of ker T is called the nullity of T.

Product of two Linear Transformations: For any two linear transformations T_1, T_2 on V we define $(T_1, T_2)(x) = T_1(T_2(x))$, $x \in V$.

Hence we call T_1, T_2 as the product of T_1 and T_2 similar $T_2 T_1$ is called the product of T_2 and T_1.

The range (or rank space) of Linear Transformation: Let T be a linear transformation of a vector space V (F) into a vector space V′ (F).

Then the set $T(V) = \{T(x) : x \in V\}$ is called the range (or rank space) of T.

Also the dim. T(V) is called the rank of the linear transformation T.

Linear Operator: A linear transformation from a vector space V (F) into V is called a linear operator.

Non Singular Linear Transformation: Let T be linear transformation on a vector space V. Then T is called invertible or non singular if T is one-one and onto otherwise T is called singular.

If T is non singular then T^{-1} exists such that $T(x) = y$ iff $x = T^{-1}(y)$

And $T T^{-1} = T^{-1} T = I$ (identity element of V).

Linear Functional (2016): Let V be a vector space over a field F, then a map $T : V \to F$ is called linear functional (L.F) iff

(i) $T(u + v) = T(u) + T(v)$ and

(ii) $T(\alpha u) = \alpha T(u), \forall \alpha \in F, u, v \in V$

Linear functional it also called linear maps or linear form.

We should remember that F can be regarded as a vector space over F itself.

5.3. Theorems:

Theorem (5.3) i :- Let T be a linear transformation form a vector space V into a vector space V′ then T preserves the origin and negative .

Observation:- Since $T(0) = T(0 \,.\, 0) = 0\, T(0) = 0$

And $T(-x) = T\{(-1).\, x\} = (-1)\, T(x) = -\, T(x),\ x \in V$

Note:- An isomorphism preserves origin and negative.

For an isomorphism is one-one, onto linear transformation.

Theorem (5.3) ii :- Let V and V′ be any two vector spaces over the same field F.

If $T : V \to V'$ is linear transformation then,

Ker T is a linear sub-space of V.

Proof :- Let $x, y \in$ Ker T then $T(x) = 0, T(y) = 0$.

Also let $\alpha, \beta \in F$ then v is a vector space $\Rightarrow \alpha x + \beta y \in V$

Also T is a linear transformation from V to V′ then $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$

$= \alpha\, .0 + \beta .0 = 0$

This implies that $\alpha x + \beta y \in$ Ker T for $x, y \in$ Ker T, $\alpha, \beta \in F$

Thus Ker T is a linear sub-space of the linear space V

Theorem (5.3) iii :- Let T be a linear transformation of a vector space V (F) into a V′(F) . Then $T(V) = \{ T(x) : x \in V \}$

Prove that T(V) is a sub-space of V′.

Proof :- Clearly $0 \in V \Rightarrow T(0) = 0' \in V' \Rightarrow T(0) \in V' \Rightarrow T(0) \Rightarrow T(0) \neq \phi$

$\Rightarrow$ T(0) is a non empty subset of V′.

Since for $x', y' \in T(V)$ there exists $x, y \in V$ such that $T(x) = x', T(y) = y'$.

Also whenever $x, y \in V$ we shall get $\alpha, \beta \in F$ such that $\alpha x + \beta y \in V$.

But T is a linear transformation as a result of which

$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) = \alpha x' + \beta y' \in T(V)$

Thus T(V) is a sub-space of V′.

Theorem (5.3) iv :- Let T be a linear transformation of a vector space V (F) into a V′(F) then map-T given by $(-T)(x) = -T(x)$, for every x in V is also a linear transformation from V into V′ .

Proof :- Since $T : V \to V'$ is a linear transformation then for x in V, $T(x) \in V'$ and V′ is a vector space $\Rightarrow -T(x) \in V'$ for $x \in V$

Again clearly for $x, y \in V$, $\alpha, \beta \in F$, $\alpha x + \beta y \in V$

Also $(-T)(\alpha x + \beta y) = -T(\alpha x + \beta y) = -[\alpha T(x) + \beta T(y)] = -\alpha T(x) - \beta T(y)$

$$= \alpha \{-T(x)\} + \beta \{-T(x)\}$$

Thus –T is also linear transformation from V into V′ .

Theorem (5.3) v :- Let $T : V \to V'$ be a linear transformation of V(F) then prove that T is one-one if and only if T is non singular

Proof :- First of all let T is non singular

To prove that T is one-one.

For, if K is the null space of T then $K = \{0\}$.

Also Let for x, y in V, $T(x) = T(y)$. ---------------(1)

Also $T(x - y) = T(x) - T(y) = T(x) - T(x)$ [by (1)]

$$= 0' = T(0)$$

Hence $x - y = 0$ and $(x - y) \in K$

$\Rightarrow x = y$

Thus $T(x) = T(y) \Rightarrow x = y = T$ is one-one

Conversely:- Let T is one-one

To prove that : T is non singular

Sufficient to show that the null space of $T = K = \{0\}$

For, let $x \in K$ be arbitrary, then $T(x) = 0'$ but $T(0) = 0'$

Thus $T(x) = T(0)$ but T is one-one so $x = 0 \Rightarrow K = \{0\}$

Thus T is non singular

Theorem (5.3) vi :- Let T_1, T_2 be any two linear transformation of V(F) into V′(F) then,

(i) $T_1 + T_2$ and (ii) αT_1

both are linear transformations

Proof :- We define $T_1 + T_2$ and αT_1 by

(i) $(T_1 + T_2)(x) = T_1(x) + T_2(x)$ and $(\alpha T_1)(x) = \alpha T_1(x)$, $x \in V$, $\alpha \in F$

Then $(T_1 + T_2)(\alpha x + \beta y) = T_1(\alpha x + \beta y) + T_2(\alpha x + \beta y) = \alpha T_1(x) + \beta T_1(y) + \alpha T_2(x) + \beta T_2(y)$

$= \alpha T_1(x) + \alpha T_2(x) + \beta T_1(y) + \beta T_2(y)$

$= \alpha (T_1(x) + T_2(x)) + \beta (T_1(y) + T_2(y))$

$= \alpha (T_1 + T_2)(x) + \beta (T_1 + T_2)(y)$

$\Rightarrow T_1 + T_2$ is linear transformation of V into V′.

(ii) $(\alpha T_1)(ax + by) = \alpha T_1(ax + by) = \alpha \{a T_1(x) + b T_1(y)\}$
$= \alpha a T_1(x) + \alpha b T_1(y) = a \{\alpha T_1(x)\} + b \{\alpha T_1(y)\}$

Thus αT_1 is also linear transformation.

Theorem (5.3) vii (2019):- If f is a linear functional on a vector space V(F) then

(i) $f(0) = 0$
(ii) $f(-x) = -f(x)$

Proof :- Since f is a linear functional so for any x in V, $f(x) \in F$

(i) since $f(x) + 0 = f(x)$ for $0 \in F$
$= f(x+0)$
$= f(x) + f(0)$

That is $f(x) + 0 = f(x) + f(0) \Rightarrow f(0) = 0$

(ii) Since $f(x) + f(-x) = 0$

$\Rightarrow f(-x) = -f(x)$

Theorem (5.3) viii (2016):- Prove that function f on R^n defined by

$f(x) = f(x_1, x_2,.....,x_n) = a_1x_1 + a_2x_2 + ...+ a_n x_n$ is function on R^n.

Where $a_1, a_2,.....,a_n$ be fixed scale of R.

Proof :- let $f : R^n \to R$ by $f(x) = f(x_1, x_2,.....,x_n) = a_1x_1 + a_2x_2 + ...+ a_n x_n$

Now since, $f\{ (x_1, x_2,.....,x_n) + (y_1, y_2,.....,y_n) \}= f(x_1+ y_1, x_2 + y_2 ,...., x_n+ y_n)$

$= a_1 (x_1+ y_1) + a_2 (x_2+ y_2) + ... + a_n (x_n+ y_n)$

$= (a_1x_1 + a_2x_2 + ...+ a_n x_n) + (a_1y_1 + a_2y_2 + ...+ a_n y_n) = f\{ (x_1, x_2,.....,x_n) +f(y_1, y_2,.....,y_n) \}$

$= f\{ x (x_1, x_2,.....,x_n) \} = f(x x_1 , x x_2,.....,x x_n) = x (a_1x_1 + a_2x_2 + ...+ a_n x_n)$

$= x f(x_1, x_2,.....,x_n)$.

Thus $x \in R$. f is a linear functional on R^n.

Theorem (5.3) ix (2018):- let $T : U \to V$ be a linear transformation then prove that

dim. ker (T) + dim. rang (T) = dim. domain (T)

Proof :- let $\{ \alpha_1, \alpha_2,....., \alpha_k \}$be a basis of ker (T) i.e, the null space of T.

Let dim. U = n then $\alpha_{k+1}, \alpha_{k+2} ,....., \alpha_n \in U$ such that $\alpha_1, \alpha_2,....., \alpha_n$ forms a basis of U.

Thus dim. ker (T) = K

Consider $\{T(\alpha_{k+1}) + T(\alpha_{k+2}) + ... + T(\alpha_n) \}$ ------------ (1)

Then α_i in F

We have $a_{k+1} T(\alpha_{k+1}) + a_{k+2} T(\alpha_{k+2}) + ... + a_n T(\alpha_n) = 0$

Then $a_{k+1} \alpha_{k+1} + a_{k+2} \alpha_{k+2} + ... + a_n \alpha_n \in \ker(T)$

Also $\{ \alpha_1, \alpha_2,....., \alpha_k \}$ is a basis of ker (T) so we can get scalars $b_1, b_2,....., b_k$

$a_{k+1} \alpha_{k+1} + a_{k+2} \alpha_{k+2} + ... + a_n \alpha_n = b_1 \alpha_1 + b_2 \alpha_2 + ... b_k \alpha_k$

or, $b_1 \alpha_1 + b_2 \alpha_2 + ... b_k \alpha_k - [a_{k+1} \alpha_{k+1} + a_{k+2} \alpha_{k+2} + ... + a_n \alpha_n] = 0$

but $\alpha_1, \alpha_2,....., \alpha_n$ are linearly independent

$\Rightarrow b_1= b_2 = b_k = a_{k+1} = ... a_n = 0$

Thus (1) is linearly independent ------------ (2)

Again, let T (α) $\in$ rang (T) for $\alpha \in$ U

Clearly $\alpha = a_1 \alpha_1 + a_2 \alpha_2 + ... + a_n \alpha_n$.

Also T (α) = T ($a_1 \alpha_1$) + T ($a_2 \alpha_2$) + ... + T ($a_n \alpha_n$) and T is a linear transformation

= a_1 T (α_1) + a_2 T (α_2) + ... + a_k T (α_k) + a_{k+1} T (α_{k+1}) + a_{k+2} T (α_{k+2}) + ... + a_n T (α_n)

Since T (α_i) = 0, $1 \le i \le k$.

Thus T (α) = a_{k+1} T (α_{k+1}) + a_{k+2} T (α_{k+2}) + ... + a_n T (α_n)

It means T (α_{k+1}), ... , T (α_n) spans rang (T) ----------- (3)

Thus from (2) and (3)

[T (α_{k+1}), ... , T (α_n)] forms a basis of spans rang (T)

Also dim. rang (T) = n – K = dim. U - dim. ker (T)

Or, dim. ker (T) + dim. rang (T) = dim. U = dim. domain (T).

6. Exercise :

Example 1:- A linear transformation T of R^3 into itself is defined by

T(e_1) = $e_1 + e_2 + e_3$; T(e_2) = $e_2 + e_3$ and T(e_3) = $e_2 - e_3$ where e_1, e_2, e_3 are unit vectors of R^3 then,

(i) Determine the transform of (2, -1, 3)

(ii) Describe explicitly the linear transformation T.

Solution:- Since e_1, e_2, e_3 are unit vectors of R^3

Thus e_1 = (1, 0, 0)

e_2 = (0, 1, 0)

e_3 = (0, 0, 1)

$\therefore$ T(e_1) = $e_1 + e_2 + e_3$ = (1, 0, 0) + (0, 1, 0) + (0, 0, 1)

$\Rightarrow$ T(e_1) = (1 + 0 + 0, 0 + 1 + 0, 0 + 0 + 1) = (1, 1, 1)

Similarly T(e_2) = (0, 1, 1)

And T(e_3) = (0, 1, -1)

We know that { e_1, e_2, e_3 } forms a basis of R^3

Thus every vector of R^3 can be uniquely expressed as the linear combination of e_1, e_2, e_3.

Clearly $(2, -1, 3) \in R^3$

Then $(2, -1, 3) = 2 (1, 0, 0) + (-1) (0, 1, 0) + 3 (0, 0, 1)$

$= 2 e_1 + (-1) e_2 + 3 e_3$

Thus $T (2, -1, 3) = T \{2 e_1 + (-1) e_2 + 3 e_3 \}$

Or $T (2, -1, 3) = 2 T(e_1) +(-1) T (e_2) + 3 T (e_3)$

$= 2 (1, 1, 1) +(-1) (0, 1, 1) + 3 (0, 1, -1)$

i.e, $T (2, -1, 3) = (2, 4, -2)$

Hence the transformation of $(2, -1, 3)$ under T is $(2, 4, -2)$.

(ii) Let (x, y, z) be any in R^3 then as we know it can be expressed as linear combination of e_1, e_2, e_3 uniquely

Thus $(x, y, z) = x (1, 0, 0) + y (0. 1, 0) + z (0, 0, 1)$

$= xe_1 + ye_2 + ze_3$

$T (x, y, z) = T (xe_1 + ye_2 + ze_3)$

$= x T(e_1) + y T(e_2) + z T(e_3)$

$= x (1, 1, 1) + y (0, 1, 1) + z (0, 1, -1) = (x, x + y, x + y – z)$

is the required linear transformation explicitly (or completely)

Example 2:- if T is a non singular linear transformation on vector space V (F) then

T^{-1} Is also a linear transformation.

Example 3:- if T is a linear transformation on vector space V (F) then

T is one-one if and only if T is onto.

Example 4:- Prove that a non singular transformation T on vector space V(F) is onto.

Example 5:- Verify that if a linear transformation T on vector space V(F) is onto then

Whether T is non singular.

Example 6:- Let (i) U, V be any two vector spaces over the same field F.

and (ii) L(U, V) be the set of all linear transformations from U to V verify that weather L(U, V) form a vector spaces over F under addition and scalar multiplication of linear transformations suitably defined.

7. Dual Space And Dual Basis:-

7.1. In this section our aim is to make a detailed study of the vector space of the linear functional.

7.2. Definitions:

Dual space of a vector space V(F):- Let V(F) be a vector space over the field F.

Clearly F can be considered as vector space over F itself.

Then vector space L(V, F) of all linear transformations of V into F is called the dual space of V (or algebraic conjugate of V)

We usually use the symbol V^* for the dual space of V.

Every element of V^* is called a linear functional on V.

Clearly $V^* = L(V, F)$. V^* is also called simply conjugate space of V.

Or, $V^* = \{ T : T : V \rightarrow F \}$.

Second dual space :- Like the vector space V, its dual space V^* has also dual space denoted by V^{**}, which is a vector space and is called second dual space of V.

Dual basis :- Let $\{ v_1, v_2,.....,v_n \}$ be a basis of the vector space V(F) .

Again let $T_1, T_2,.....,T_n \in V^*$ be linear functional defined by :

$T_i (v_J) = \Psi i_J = \{ 1$ if $i = J$ & 0 if $i \neq J$

Then a basis $\{ T_1, T_2,.....,T_n \}$ of V is called dual basis

7.3. Theorems:

Theorem (7.3) i :- Let V(F) be a finite dimensional vector space then

Then prove that : dim. $V^* =$ dim. V

Proof :- let V(F) be a finite dimensional vector space such that

$$\text{dim. } V = n$$

again let V^* be the dual space of V.

To prove that dim. $V^* = n$

Since V^* is the set of all linear functional on V(F)

i.e, $V^* = L(V, F)$

Since we know by a theorem that if L(U, V) is the set of all linear transformations from U(F) into V(F) then

dim. L(U, V) = (dim. U) (dim. V)

Thus dim. V^* = dim. (V, F) = (dim. V) (dim. F) = n . 1 = n = dim. V

Thus dim. V^* = dim. V.

Theorem (7.3) ii :- Let { $v_1, v_2, \ldots, v_n$ } be a basis of the vector space V(F), over the field F

Also let $T_1, T_2, \ldots, T_n \in V^*$ be the linear functional defined by

$T_i (v_J) = \{ 1$ if $i = J$ & 0 if $i \neq J$ ---------------- (1)

Then { $T_1, T_2, \ldots, T_n$ } is a basis of V^*

i.e, { $T_1, T_2, \ldots, T_n$ } is a dual basis of the basis { $v_1, v_2, \ldots, v_n$ }

Proof :- First of all we make efforts to show that the set { $T_1, T_2, \ldots, T_n$ } generates V^*.

For this, let $T(v_i) = t_i$, for $1 \leq i \leq n$ and $t_1 T_1 + t_2 T_2 + \ldots + t_n T_n = P$ ----------(2)

Then $T \in V^*$ and $P(v_1) = \{ t_1 T_1 + t_2 T_2 + \ldots + t_n T_n \} (v_1) = t_1$ [By (1)]

Let i = 2, 3, ... , n ; then $P(v_i) = \{ t_1 T_1 + t_2 T_2 + \ldots + t_n T_n \} (v_i) = t_i$ [By (1)]

Thus $P(v_i) = t_i$, $1 \leq i \leq n$

But $T(v_i) = t_i$, $1 \leq i \leq n$

It means P = T for every v_i

But by (2) P and hence T is generated by the set { $T_1, T_2, \ldots, T_n$ }

Thus the set { $T_1, T_2, \ldots, T_n$ } generates V^*----------(3)

We now make efforts to show that the set { $T_1, T_2, \ldots, T_n$ } is L.I.

For, $a_1 T_1 + a_2 T_2 + \ldots + a_n T_n = 0$ then $(a_1 T_1 + a_2 T_2 + \ldots + a_n T_n)(v_r) = 0 (v_r)$

On simplification we get $a_r = 0$, $1 \leq r \leq n$.

Thus $a_1 T_1 + a_2 T_2 + \ldots + a_n T_n = 0 \Rightarrow a_1 = 0, a_2 = 0, \ldots, a_n = 0$

Thus the set { $T_1, T_2, \ldots, T_n$ } is linearly independent ------------ (4)

Thus from (3) and (4)

The set { $T_1, T_2, \ldots, T_n$ } forms a basis of V^*

Theorem (7.3) iii (2016):- Let (i) V(F) be a finite dimensional vector space over the field F.

(ii) B be a basis of V (iii) B′ be a dual basis of V

Then to show that B″ = (B′)′ = B

Observation :- Let B = { $x_1, x_2, \ldots, x_n$ } be a basis of V

And B′ = { $f_1, f_2, \ldots, f_n$ } be a basis of V^*

B″ = { $e_1, e_2, \ldots, e_n$ } be a basis of V^{**},

By question V(F) is a finite dimensional vector space

Let dim. V = n

But we know that dim. V = dim. V^* = dim. V^{**} = n

We also know that dim. (V) = the number of elements in its basi B.

Thus clearly each basis B, B′ and B″ will have the same number of elements.

Since $f_1 (x_J) = \delta i_J$ ---------(1)

and $e_i (f_J) = \delta i_J$ ---------(2)

now for $x \in V$ we get e_x in V^{**} such that $e_x (f) = f(x)$, $f \in V^*$ ---------(3)

then $e_{xi} (f_i) = f(x_i) \Rightarrow e_{xi} (f_J) = f_J (x_i) = \delta i_J = e_i (f_J)$ [by (1) and (2)]

for J = 1, 2, ... , n

$\Rightarrow e_{xi} = e_i$

Thus by natural isomorphism $x \to e_x$, e_x is same as x.

$e_i = e_{xi} = x_i \Rightarrow$ B″ = B

Solved examples :-

Example 1:- Let $V_3(R)$ be a vector space and B = { (1, -1, 3), (0, 1, -1), (0, 3, -2) }be a basis of $V_3(R)$. Find its dual basis .

Solution:- Let B = { x_1, x_2, x_3 } be a basis of V and its dual basis be

$B' = \{ f_1, f_2, f_3 \}$ where

$f_1 (x, y, z) = a_1 x + a_2 y + a_3 z \quad a_1, a_2, a_3 \in R$

$f_2 (x, y, z) = b_1 x + b_2 y + b_3 z$

$f_3 (x, y, z) = c_1 x + c_2 y + c_3 z$

Now $f_1 (x_1) = f_1 (1, -1, 3) = a_1 - a_2 + 3 a_3 = 1$

Similarly $f_1 (x_2) = f_1 (0, 1, -1) = 0 + a_2 - a_3 = 0$

$f_1 (x_3) = f_1 (0, 3, -2) = 0 + 3 a_2 – 2 a_3 = 0$

By solving we get $a_1 = 1, a_2 = 0, a_3 = 0$

Thus $f_1 (x, y, z) = 1 . x + 0 . y + 0 . z = x$ -------- (1)

lly $f_2 (x_1) = 0 \Rightarrow b_1 - b_2 + 3 b_3 = 0$

$f_2 (x_2) = 1 \Rightarrow 0 + b_2 - b_3 = 1$

$f_2 (x_3) = 0 \Rightarrow 0 + 3b_2 - 2b_3 = 0$

On solving we get $b_1 = 7, b_2 = -2, b_3 = -3$

Thus $f_2 (x, y, z) = 7x – 2y – 3z$ ------------(2)

Finally $f_3 (x_1) = 0, f_3 (x_2) = 0, f_3 (x_3) = 1$

On solving we get $c_1 = -3, c_2 = 1, c_3 = 1$

Thus $f_3 (x, y, z) = -2x + y + z$ --------------(3)

Thus dual basis $B' = \{ f_1, f_2, f_3 \}$

$= \{x, 7x– 2y – 3z, -2x + y + z\}$

Example 2:- If $B = \{ e_1, e_2, e_3 \}$ be a basis of $R^3(R)$ then find dual basis B'.

Solution:- Since e_1, e_2, e_3 are unit vectors in R^3 so we have

$e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$.

For a moment let $B = \{ e_1, e_2, e_3 \}= \{ x_1, x_2, x_3 \}; B' = \{ f_1, f_2, f_3 \}$

Let $f_1 (x, y, z) = a_1 y + a_2 y + a_3 z$

$f_2 (x, y, z) = b_1 y + b_2 y + b_3 z$

$f_3 (x, y, z) = c_1 y + c_2 y + c_3 z$

Where $a_i, b_i, c_i \in R$, i = 1, 2, 3.

Now $f_1(x_1) = 1 = f_1(e_1) = f_1(1, 0, 0) = a_1$

$f_1(x_2) = 0 = f_1(e_2) = f_1(0, 1, 0) = a_2$

$f_1(x_3) = 0 = f_1(e_3) = f_1(0, 0, 1) = a_3$

Hence $f_1(x, y, z) = a_1 x + a_2 y + a_3 z = x$

Also $f_2(e_1) = 0$, $f_2(e_2) = 1$, $f_1(e_3) = 0$

And $f_3(e_1) = 0$, $f_3(e_2) = 0$, $f_3(e_3) = 1$

Thus $f_2(x, y, z) = y$,

$f_3(x, y, z) = z$

Thus B′ = { x, y, z }.

Example 3:- Let $f : R^3 \rightarrow R$, $g : R^3 \rightarrow R$ be two linear functions given by

$f(x, y, z) = 2x – y + z$ and

$g(x, y, z) = 3x – y + 2z$, then

Find:- (1) 3f (2) 5g (3) 4f – 5g

Solution:- (1) Since $3f = 3(2x – y + z) = 6x – 3y + 3z$

(2) Since $5g = 5(3x – y + 2z) = 15x – 5y + 10z$

(3) $4f – 5g = -7x + y – 6z$

Example 4:- If B = { (-1, 1, 1), (1, -1, 1), (1, 1, -1) } is a basis of $V_3(R)$, then find the dual basis B′ of B.

Solution:- Do as above.

8. Projection

8.1 Introduction:
Projection is in fact a particularly important type of linear transformation on a linear space over some field.

8.2 Definition:

Idempotent: A linear transformation T of a vector space V(F) into V is called an idempotent if $T^2 = T$. Also T is called nilpotent if $T^2 = 0$.

Invariant: Let T be a linear operator on a vector space V(F) and W be its a subspace. Then we say that W is invariant under T if T maps W into W itself. W is also called T – invariant.

Thus if any $x \in W \Rightarrow T(x) \in W \Rightarrow T(W) \subseteq W$

Projection: Let L be the direct sum of the subspaces M and N so that, $L = M \oplus N$. Then each vector z in L can be written uniquely in the form $z = x + y$ with x in M and y in N. As x is uniquely determined by z so we can define a mapping E of L into itself be $E(z) = x$. Then E is called the projection on M along N.

8.2 Theorems:

Theorem (8.3) i :- Projection E is a linear transformation.

Proof :- Let $l_1, l_2 \in L$ such that $l_1 = m_1 + n_1$, $l_2 = m_2 + n_2$ ---------------(1)

where $m_1, m_2 \in M$, $n_1, n_2 \in N$

Then $\alpha l_1 + \beta l_2 = \alpha (m_1 + n_1) + \beta (m_2 + n_2) = (\alpha m_1 + \beta m_2) + (\alpha n_1 + \beta n_2)$

Since M, N are subspaces so clearly $\alpha m_1 + \beta m_2 \in M$, $\alpha n_1 + \beta n_2 \in N$ and E is a projection.

Hence $E (\alpha l_1 + \beta l_2) = E [(\alpha m_1 + \beta m_2)+(\alpha n_1 + \beta n_2)]$

$= \alpha m_1 + \beta m_2$ -----------(2) [Since if $z = x + y$ then $E(z) = x$]

Also from (1) $E (l_1) = m_1$, $E (l_2) = m_2$ (By definition of projection)

Thus from (2) we have

$E (\alpha l_1 + \beta l_2) = \alpha m_1 + \beta m_2 = \alpha E (l_1) + \beta E (l_2)$

Thus E is a linear transformation of L into L itself.

Corollary 1: $E (m) = m \Leftrightarrow m \in M$ and $E(n) = 0 \Leftrightarrow n \in N$.

For $m \in M \Rightarrow E (m + 0) = m$

$n \in N \Rightarrow E (0 + n) = 0$

Theorem (8.3-2018, 2019) ii :- A linear transformation E is a projection on some subspace if and only if it is idempotent. That is $E^2 = E$.

Proof :- Let a vector space L is the direct sum of its two subspaces M and N.

That $L = M \oplus N \Rightarrow l$ in L can be uniquely expressed as

$l = m + n$ for $m \in M$, $n \in N$

Let E be the projection on M along N then E (l) = E (m + n) = m

Also E (m) = m.

To prove that E is idempotent i.e, $E^2 = E$.

For this,

Since E^2 (l) = E E(l) = E [E(l)] = E (m) [Since E(l) = m ⇒ E [E(l)] = E (m)]

= m = E(l) [by corollary]

⇒ E^2(l) = E(l) ⇒ $E^2 = E$ ⇒ E is idempotent.

Conversely let E be a linear transformation on L and $E^2 = E$ i.e E is an idempotent.

To prove that E is a projection on some subspace

For , let U = { l ∈ L : E(l) = E } ---------------(1)

And W = { l ∈ L : E(l) = 0 } -------------------(2)

Then we observe that:

Since 0 ∈ L and E(0) = 0 and hence 0 ∈ U, 0 ∈ W.

Now let α, β ∈ F (the field), $u_1, u_2 \in U$, $w_1, w_2 \in W$.

Then $E(u_1) = u_1$, $E(u_2) = u_2$, $E(w_1) = 0 = E(w_2)$

Also by assumption, E is linear and hence

$E(a\, u_1 + b\, u_2) = a\, E(u_1) + b\, E(u_2) = a\, u_1 + b\, u_2$

Also $a\, u_1 + b\, u_2 \in U$ for every $u_1, u_2 \in U$

Also 0 ∈ U, as a result of which U is a subspace of L(F)

Also $E(a\, w_1 + b\, w_2) = a\, E(w_1) + b\, E(w_2)$

= a.0 + b.0 = 0 [since $w_1, w_2 \in W$]

Thus $E(a\, w_1 + b\, w_2) = 0 \Rightarrow a\, w_1 + b\, w_2 \in W$.

Also 0 ∈ W then W is a subspace of L(F)

Thus U and W are subspace of L(F)

Further :

Let $l \in L$ then $l = 0 + l = E(l) - E(l) + I(l)$ [Since $E(l) - E(l) = 0$ and $I(l) = l$]

$= E(l) + (I - E)(l) \Rightarrow l = u + w$.

Also $E(u) = E\,E(l) = E^2(l) = E(l) = u$ (since $E^2 = E$ by assumption)

$E(w) = E(I - E)(l) = (El - E^2)(l) = (E - E)l = 0$

Thus $E(u) = u$, $E(w) = 0$

Thus from above we find that

$l = u + w$ i.e. l is the linear sum of u and w.

Also, let $x \in u \cap w \Rightarrow x \in u$ and $x \in w \Rightarrow E(x) = x, E(x) = 0 \Rightarrow x = E(x) = 0 \Rightarrow x = 0$

Thus $u \cap w = \{0\}$ and from above $L = U + W \Rightarrow L = U \oplus W$.

Finally $E(l) = E(u + w) = E(u) + E(w) = u + 0 = u$

Thus E satisfied all the conditions to be a projection on U and W.

Theorem (8.3) iii :- If $V = w_1 \oplus w_2 \oplus\text{--------}\oplus w_n$, then there exist n-linear operators E_1, $E_2, \ldots, E_n$ such that

(i) each E_i is a projection:- i.e, $E_i^2 = E_i$ for every i
(ii) $E_i E_j = 0$ if $i \neq j$
(iii) $E_1 + E_2 + \ldots + E_n = I$
(iv) Rang $(E_i) = W_i$

Proof :-

Part (i) : To prove E_i is a projection; i.e. $E_i^2 = E_i$ for every i.

For let $V = w_1 \oplus w_2 \oplus\text{--------}\oplus w_n$, then $\alpha, \beta \in V$ can be uniquely expressed as

$\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_n$, $\beta = \beta_1 + \beta_2 + \ldots + \beta_n$ such that $\alpha_i, \beta_i \in W_i$ for each i

We now define a function $E_j : V \to V$ such that $E_j(\alpha) = E_j(\alpha_1 + \alpha_2 + \ldots + \alpha_n) = \alpha_j$

Let $a, b \in F$ then $a\alpha_i, b\beta_i \in W_i$ (as each W_i is a subspace of V)

Clearly $a\alpha, b\beta \in V$ for $\alpha, \beta \in V$ (as V is a linear space)

Also $a\alpha + b\beta = a(\alpha_1 + \alpha_2 + \ldots + \alpha_n) + b(\beta_1 + \beta_2 + \ldots + \beta_n) = (a\alpha_1 + b\beta_1) + \ldots + (a\alpha_n + b\beta_n)$

So $E_j(a\alpha + b\beta) = a\alpha_j + b\beta_j = a E_j(\alpha) + b E_j(\beta) \Rightarrow E_j$ is a linear map.

Also we know by definition that ,

$E_j (\alpha_1 + ...+\alpha_j + \alpha_{j+1} + ...+\alpha_n) = \alpha_j \Rightarrow E_j (0 + 0 + ... + \alpha_j + 0 + ...+ 0) = \alpha_j$

Then $E_j (\alpha_j) = \alpha_j$ and hence ,

$E_i (\alpha_1 + ...+\alpha_i + ...+\alpha_n) = \alpha_i$ then $E_i (0 + 0 + ... + \alpha_i + 0 + ...+ 0) = \alpha_i$

That is, $E_i (\alpha) = \alpha_i = E_i(\alpha_j)$ -------------(1)

Now $E_i^2 (\alpha) = (E_i E_i) (\alpha) = E_i [E_i(\alpha)] = E_i(\alpha_i)$ (by (1))

$= \alpha_i = E_i(\alpha)$

That is $E_i^2 (\alpha) = E_i (\alpha) \Rightarrow E_i^2 = E_i$ and we seen that E_i is a linear map

Therefore E_i is a projection.

Part (ii) : Let $i \neq j$ then $(E_i E_j) = E_i[E_j(\alpha)] = E_i(\alpha_j)$ (from above)

$$= 0 = 0 (\alpha)$$

i.e. $(E_i E_j) (\alpha) = 0 (\alpha)$

Thus $E_i E_j = 0$

Part (iii) : Since $(E_1 + E_2 + ...,+E_n) (\alpha) = (E_1 (\alpha) + E_2(\alpha) + ...+ E_n(\alpha))$

$$= \alpha = I(\alpha) \quad [\text{see } (1)]$$

i.e. $(E_1 + E_2 + ...,+E_n) (\alpha) = I(\alpha) \Rightarrow E_1 + E_2 + ...,+E_n = I$.

Part (iv) : Since Rang $(E_i) = \{ E_i (\alpha) : \alpha \in V\}$

$\alpha_i \in W_i \Rightarrow E_i (\alpha_i) = \alpha_i$ (By definition of E_i)

$\Rightarrow \alpha_i \in$ Rang (E_i), for $E_i (\alpha_i) \in$ Rang $(E_i) \Rightarrow W_i \subset (E_i)$

Now if any $x \in$ Rang $(E_i) \Rightarrow x \in V$ them we can get $y \in V$ such that $E : (y) = x$

$\Rightarrow E_i (y_1 + y_2 + ...+y_n) = x$ where $y_1 + y_2 + ...+y_n = y$ and $y_i \in W_i$ for $i = 1, 2, ..., n$.

$\Rightarrow y_i = x$, and $y_i \in W_i$ so $x \in W_i$ [since by (1) $E_i (y_i) = y_i$]

Thus we get $x \in$ Rang $(E_i) = x \in W_i \Rightarrow$ Rang $(E_i) \subseteq W_i$ -----------(2)

But $W_i \subseteq$ Rang (E_i) --------------(3)

Thus from (2) and (3) and the definition of the equality of any two sets Rang $(E_i) = W_i$ //

Theorem (8.3) iv :- Let V be the direct sum of its subspaces U and W and E is the projection on U along W then I – E is a projection on W along U.

Proof :- Since E is the projection on U along W it means that $V = U \oplus W$

Such that , U = the rang space of E and W = the null space of E

Let $x \in U \Rightarrow x = E(y)$ for $y \in V$

$\Rightarrow (I - E)(x) = x - E(x) = E(y) - E(y) = 0$

$\Rightarrow x \in$ Null space of I – E.

Also $x \in W \Rightarrow E(x) = 0 \Rightarrow (I - E)(x) = x$ for every x in W

Thus $v \in V \Rightarrow v = u + w, u \in U, w \in W$

$\Rightarrow (I - E) v = (I - E) u + (I - E) w = 0 + w = w$

The rang space of (I – E) = w

Also $(I - E)^2 = I^2 + E^2 - 2 IE = I + E - 2E = (I - E)$

That is $(I - E)^2 = (I - E) \Rightarrow (I - E)$ is an idempotent.

Thus (I – E) is a projection on W along U.

Theorem (8.3) v :- If E is a projection, then its ad joint E^* is also a projection.

Proof :- Let E be a projection, then $E^2 = E$.

We have to show that E^* is also a projection.

For this it is sufficient to show that $(E^*)^2 = E^*$

For this, since $E^2 = E \Rightarrow EE = E \Rightarrow (E E)^* = E^* \Rightarrow E^* E^* = E^* \Rightarrow (E^*)^2 = E^*$

$\Rightarrow E^*$ is idempotent and hence a projection.//

Solved problems :-

Problem 1:- Let V be a real vector space and E an idempotent linear operator. Prove that I + E is invertible.

Solution:- Since $(I + E)(I - \frac{1}{2} E) = I + E - \frac{1}{2} E - \frac{1}{2} E^2 = I + E - \frac{1}{2} E - \frac{1}{2} E = I + E - E = I$

Hence I + E is invertible and $(I + E)^{-1} = I - \frac{1}{2} E$.

Problem 2:- If E and F are projection on a vector space V(K), then prove that E + F – EF is a projection provided EF = FE.

Solution:- By question E, F are projections on V(F)

Then E, F are idempotent, but then $E^2 = E$, $F^2 = F$

Let (as the provision is) E and F commute i.e. EF = FE.

Our problem is to establish that E + F – EF is a projection.

For this it is sufficient to show that E + F – EF is an idempotent.

For thus it is sufficient to show that $(E + F – EF)^2 = E + F – EF$

For this, L.H.S = $(E + F – EF)^2$ =(E + F – EF)(E + F – EF) Then by actual multiplication

$= (E^2 + EF - E^2 F)+ (FE+ F^2– FEF) +(-EFE – E F^2 + EFEF)$

$=(E + EF – EF) + (EF + F – EFF) + (- EFE - E F +EFFE)$

$= (E + 0) +(EF + F - EF^2) + (-E^2 F – EF + EF^2E)$

$=E + F + (- EF – EF + EEF)$

$= E + F + (- EF – EF + EF)$

i.e. $(E + F – EF)^2 = E + F – EF$

Hence our requirement is obtained.

Problem 3:- If T is a linear operator on a vector space V(K) such that

$T^2(I – T) = T (I – T)^2 = 0$, then prove that T is a projection.

Solution:- To prove T is a projection.

It is sufficient to prove T is idempotent i.e. $T^2 = T$

For this we are given that T is a linear operator on a vector space V(K),such that,

$T^2(I – T) = T (I – T)^2 = 0$

$\Rightarrow T^2 I – T^3 = T (I^2+ T^2 -2 IT) = 0$

$\Rightarrow T^2 – T^3 = T (I+ T^2 -2 T) = 0$

$\Rightarrow T^2 – T^3 = T I+ T^3 - 2 T^2 = 0$

$\Rightarrow T^2 - T^3 = 0$ and $T + T^3 -2 T^2 = 0$

$\Rightarrow T^2 = T^3$ -----------(1) and $T^3 = 2\,T^2 - T$ -----------(2)

Thus from (1) and (2)

$T^2 = 2\,T^2 - T \Rightarrow -\,T^2 = -T$

$\Rightarrow T^2 = T \Rightarrow$ T is idempotent $\Rightarrow$ T is a projection/

Problem 3:- Let E be a projection on a subspace U of a vector space V(K). Then V is T-invariant if and only if E T E = T E. T being a linear operator on V.

Solution:- Do yourself..